An optical soliton pair among absorbing three-level atoms

Hichem Eleuch¹ and Raouf Bennaceur²

¹ Institut National des Sciences Appliquées et de Technologie, Zone Urbaine Nord Tunis, BP 676, Tunis Cedex 1080, Tunisia
² Laboratoire de Physique de la Matière Condensée, Département de Physique, Faculté des Sciences de Tunis, Campus Universitaire, 2092 El Manar 1, Tunis, Tunisia

E-mail: hichem.eleuch@insat.rnu.tn and raouf.bennaceur@planet.tn

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Abstract

The motion of a pair of solitons propagating through an absorbing three-level system in the lambda configuration is analysed. In the course of this analytic analysis, the existence condition for solitary wave pairs is derived. This condition allows for two possible velocities of the soliton pair.

Keywords: Nonlinear optics, soliton pair propagation, three-level atoms, two-photon resonance

1. Introduction

The study of resonance fluorescence of atoms has proved to provide a successful route to the understanding of matter–radiation coupling. Real atoms are complicated systems, and even the simplest real atom, the hydrogen atom, has non-trivial energy level structure. Usually, the interaction between an electromagnetic field and atoms is modelled by an interaction between the field and a quantum system with only two proper energy states [1, 2]. Using this theoretical approximation, interesting phenomena have been predicted and then observed, such as the resonance fluorescence spectrum [3, 4], photon antibunching [5–7], sub-Poissonian photon statistics [8, 9], squeezing [10–12], photon echoes [13], and self-induced transparency [14–16]. With multilevel treatments of atoms, other phenomena are explained, such as quantum jumps [17–20] and laser cooling and trapping [21, 22]

An optically thin ensemble of three-level atoms in the lambda configuration becomes perfectly transparent when driven by two independent laser fields in two-photon resonance [23–26]. This is of especial importance in spectroscopic applications since, under these conditions, spontaneous emission from the upper level is completely absent. The phenomena are understood as a destructive quantum interference between the two branches of the transition schema. This occurs under stationary operating conditions. Experimentally, it has been shown that in the dynamical excitation of three-level atoms, the medium can be made transparent [27, 28].

Our work consists of a study of the propagation of fields in optically thick samples (we consider practically unlimited media and we neglect the boundary condition problems) where the amplitudes of the fields become dynamic variables themselves. We look for the analytic expression governing a pair of solitons which propagates through an absorbing three-level lambda system. Previous analytic studies [32–34] of soliton pairs propagating in three-level systems assumed that the media are lossless. In this paper, we study a more realistic physical case, taking into account the radiative decay rates of the media.

In the next section, we present the model and the basic equations for the dynamics of three-level systems. Section 3 is devoted to the study of the analytic shapes for a soliton pair propagating in a three-level medium. In section 4, we examine the properties of this soliton pair, and from these we deduce the existence condition. Under this condition, the soliton pair propagates through the medium at two possible velocities. Finally, we conclude with possible uses of solitons in optical data communication.

2. Model

Let us consider a three-level system in the lambda configuration (the three-level atom) interacting with two resonant electromagnetic fields. The medium is excited by two laser fields applied to the Stokes and pump transitions (figure 1).

The three-level atom is described by a quantum system with three energy levels, [0], [1], and [2], taking into account the rates γ₁,₂ of radiative decay from the higher level [0] to the levels [1] and [2], and neglecting the other dissipation effects.
This three-level system is irradiated by a light beam propagating along an arbitrary direction \( x \), with polarization adequate to couple the two optical transitions, and containing two monochromatic fields, each of which is close to resonance with one of the transitions. This light beam is classically described as follows:

\[
E'(x, t) = E_1(x, t) + E_2(x, t)
\]

where \( k_1 \) and \( k_2 \) are the wavenumbers:

\[
k_j = \frac{\omega_j}{c}
\]

and \( c \) is the speed of light in a vacuum.

\( E_1 \) and \( E_2 \) are the amplitudes of the two waves and are assumed to be slowly varying functions of \( x \) and \( t \) in the following sense [2, 29, 31]:

\[
\left\lfloor \frac{1}{\omega_j} \frac{\partial \tilde{E}_j}{\partial t} \right\rfloor \ll |\tilde{E}_j|,
\]

\[
\left\lfloor \frac{e}{\omega_j} \frac{\partial \tilde{E}_j}{\partial x} \right\rfloor \ll |\tilde{E}_j|.
\]

The Hamiltonian describing the interaction of the three-level atom with the two fields has the expression

\[
H = H_0 + H_1 + H_2.
\]

The first term of the Hamiltonian \( H_0 \) corresponds to the proper energies of the atom:

\[
H_0 = \sum_{i=0}^{2} \varepsilon_i |a_i^*a_i\rangle
\]

where \( a_i, a_i^* \) are respectively the annihilation and creation fermion operators for the atomic level \( i \), and \( \varepsilon_i \) is the energy of the level \( i \). \( a_i, a_i^* \) obey the anticommutation relation

\[
[a_i, a_j^*] = \delta_{ij}.
\]

The second \( (H_1) \) and third \( (H_2) \) terms of the Hamiltonian describe the interaction between the two fields and the atom [2]:

\[
H_1 = g_1 (a_0^*a_1E_1 + a_1^*a_0E_1^*),
\]

\[
H_2 = g_2 (a_0^*a_2E_2 + a_2^*a_0E_2^*).
\]

The two dipole transition matrix elements which are assumed to be real are denoted by \( g_1 \) and \( g_2 \).

To study the evolution of the interaction between the atom and the fields, we use the density matrix formalism. The three-level density matrix equation of motion is

\[
\frac{d}{dt} \rho = \frac{1}{i\hbar} [H, \rho] + \frac{d}{dt} \rho_{\text{irr}}
\]

where \( \frac{d}{dt} \rho_{\text{irr}} \) describes the dissipation in the total system and can be modelled by the interaction of the total system with the thermal reservoir at the temperature \( 0 \) K of the thermal bath. \( \frac{d}{dt} \rho_{\text{irr}} \) has the following expression:

\[
\frac{d}{dt} \rho_{\text{irr}} = \frac{\gamma_1}{2} [a^*_1a_1, \rho] + \frac{\gamma_1}{2} [a^*_1a_1, \rho] + \frac{\gamma_2}{2} [a^*_2a_2, \rho] + \frac{\gamma_2}{2} [a^*_2a_2, \rho].
\]

In terms of the basis set of the bare atom \( |0\rangle, |1\rangle, |2\rangle \), we have

\[
a_i^*a_j = |i\rangle \langle j|,
\]

\[
\rho = \sum_{i,j=0}^{2} |i\rangle \langle j|.
\]

The expression of the total Hamiltonian of the system becomes, in this basis,

\[
H = \sum_{i=0}^{2} \varepsilon_i |i\rangle \langle i| + g_1 (E_1 |0\rangle \langle 1| + |1\rangle \langle 0|E_1^* + g_2 (E_2 |0\rangle \langle 2| + |2\rangle \langle 0|E_2^*).
\]

So, we obtain the equations of motion of the density matrix in this basis:

\[
\frac{d}{dt} \rho_{10} = i\omega_{10} \rho_{10} - i\gamma_1 (\rho_{10} - \rho_{11})
\]

\[
+ \frac{i}{2} d_{21} E_2^* \rho_{10} - \frac{\gamma_1 + \gamma_2}{2} \rho_{10},
\]

\[
\frac{d}{dt} \rho_{20} = i\omega_{20} \rho_{20} - i\gamma_2 (\rho_{20} - \rho_{22})
\]

\[
+ \frac{i}{2} d_{12} E_1^* \rho_{20} - \frac{\gamma_1 + \gamma_2}{2} \rho_{20},
\]

\[
\frac{d}{dt} \rho_{11} = i\omega_{11} \rho_{11} - i\gamma_1 (\rho_{11} - \rho_{10})
\]

\[
+ \frac{i}{2} d_{21} E_2^* \rho_{11} - \frac{\gamma_1 + \gamma_2}{2} \rho_{11},
\]

\[
\frac{d}{dt} \rho_{22} = i\omega_{22} \rho_{22} - i\gamma_2 (\rho_{22} - \rho_{20})
\]

\[
+ \frac{i}{2} d_{12} E_1^* \rho_{22} - \frac{\gamma_1 + \gamma_2}{2} \rho_{22},
\]

\[
\frac{d}{dt} \rho_{10} = \frac{d}{dt} (\rho_{10})^* \quad \text{for } i, j = 0, 1, 2
\]

where \( \omega_{10} \) and \( \omega_{21} \) are the two atomic transition frequencies:

\[
\omega_{10} = \frac{\varepsilon_0 - \varepsilon_1}{\hbar},
\]

\[
\omega_{20} = \frac{\varepsilon_0 - \varepsilon_2}{\hbar},
\]

and \( d_i = g_i / \hbar \) are the coupling constants.

From the property of the density matrix \( \text{tr}(\rho) = 1 \) we obtain

\[
\frac{d}{dt} \rho_{00} = -\frac{d}{dt} (\rho_{11} + \rho_{22}).
\]

The diagonal elements of the density matrix \( \rho \) describe level populations and determine the internal energy of the atom. The in-diagonal elements describe atomic coherences. The \( \rho_{10} \) and \( \rho_{20} \) terms oscillate at the respective driving field frequency and the \( \rho_{21} \) term oscillates with the frequency differences of the two light fields. So, we can define the slowly varying amplitudes.
of the off-diagonal density matrix elements $\tilde{\rho}_{10}$, $\tilde{\rho}_{20}$, and $\tilde{\rho}_{21}$ through the relations

\begin{align}
\rho_{j0} &= \rho_{j0} \exp(i\omega_{j0}t) & j &= 1, 2 \\
\rho_{21} &= \tilde{\rho}_{21} \exp(i(\omega_{20} - \omega_{10})t).
\end{align}

We decompose the off-diagonal elements into an imaginary part and a real part:

\begin{align}
\tilde{\rho}_{j0} &= U_{j0} + iV_{j0} \\
\tilde{\rho}_{21} &= U_{21} + iV_{21}.
\end{align}

The Hermitian property of the density matrix ensures that the diagonal elements $\rho_{11}$, $\rho_{22}$, and $\rho_{00}$ must be real.

We limit ourselves in this paper to the study of the two-photon resonance where the two laser fields $E_1$ and $E_2$ and the two transitions of the three-level system ($|0\rangle \leftrightarrow |1\rangle$ and $|0\rangle \leftrightarrow |2\rangle$) are respectively in resonance ($\omega_{10} = \omega_1$ and $\omega_{20} = \omega_2$).

In this case, the system of evolution equations of the density matrix become

\begin{align}
\frac{d}{dt} q_{jj} &= -2\Lambda_j V_{j0} + \gamma_j q_{00} & j &= 1, 2 \\
\frac{d}{dt} q_{00} &= -\frac{d}{dt}(q_{11} + \rho_{22}) \\
\frac{d}{dt} U_{10} &= \Lambda_2 V_{21} - \left(\frac{\gamma_1 + \gamma_2}{2}\right) U_{10} \\
\frac{d}{dt} U_{20} &= -V_{21} \Lambda_1 - \left(\frac{\gamma_1 + \gamma_2}{2}\right) U_{20} \\
\frac{d}{dt} V_{10} &= -i\Lambda_1 (\rho_{00} - \rho_{11}) + U_{21} \Lambda_2 - \left(\frac{\gamma_1 + \gamma_2}{2}\right) V_{10} \\
\frac{d}{dt} V_{20} &= -i\Lambda_2 (\rho_{00} - \rho_{22}) + U_{21} \Lambda_1 - \left(\frac{\gamma_1 + \gamma_2}{2}\right) V_{20} \\
\frac{d}{dt} U_{21} &= -\Lambda_1 V_{20} - V_{10} \Lambda_2 \\
\frac{d}{dt} V_{21} &= \Lambda_1 U_{20} - U_{10} \Lambda_2
\end{align}

where $\Lambda_{1,2}$ are Rabi frequencies:

$$\Lambda_j = d_j \tilde{E}_j = \frac{g_j}{\hbar} \tilde{E}_j.$$  

In this section, we have developed the evolution of the atomic parameters. In the next section, we study the propagation of fields through the medium. So, we must study the spatial and temporal evolution of the fields. The fields are classically described. Maxwell’s wave equation is needed to describe the evolution of the light pulses.

### 3. A soliton pair in a three-level medium

We study in this section three-level atoms excited by two laser fields $E_1$, $E_2$ applied on the Stokes and pump transitions respectively and we look for the pair of soliton shapes propagating through this medium.

The Maxwell equations which describe the wave evolution for the slowly varying wave amplitudes are [29, 30]

\begin{align}
\frac{\partial \tilde{E}_j}{\partial t} + c \frac{\partial \tilde{E}_j}{\partial x} &= ig_j \tilde{\rho}_{j0}.
\end{align}

where $c$ is the light velocity, $g_{1,2}'$ are the propagation constants, and $\tilde{E}_{1,2}$ are the envelopes of the two fields. $g_{1,2}'$ and $\tilde{E}_{1,2}$ are assumed to be real.

Our problem is that of finding the pair of solitons propagated through the medium. So, we can write

$$\tilde{E}_j(x, t) = \tilde{E}_j(x - vt)$$

where $v_1$ and $v_2$ can be interpreted as the soliton velocities through the medium. We assume that the two solitons propagate at the same velocity: $v = v_1 = v_2$. Hence, we introduce a new variable $z$:

$$z = x - vt$$

and we have

$$\frac{\partial}{\partial t} = -v \frac{\partial}{\partial z}$$

We assume that the two spontaneous emission rates are approximately equal values: $\gamma_1 = \gamma_2 = \gamma$.

The fact that $\tilde{E}_1$ and $\tilde{E}_i$ are real gives us

$$U_{20} = V_{21} = U_{10} = 0.$$  

The complete set of evolution equations for the medium–fields interaction (Maxwell–Bloch equations) can be obtained from the Maxwell equations and the system of evolution equations for the density matrix:

\begin{align}
\frac{\partial}{\partial z} q_{jj} &= 2\alpha_j V_{j0} - \Gamma q_{00} \\
\frac{\partial}{\partial z} \alpha_j &= -g_{E_j} V_{j0} \\
\frac{\partial}{\partial z} V_{10} &= \alpha_1 (1 - q_{22} - 2\rho_{11}) - \alpha_2 U_{21} + \Gamma V_{10} \\
\frac{\partial}{\partial z} V_{20} &= \alpha_2 (1 - q_{22} - 2\rho_{11}) - \alpha_1 U_{21} + \Gamma V_{20} \\
\frac{\partial}{\partial z} U_{21} &= \alpha_1 V_{20} + \alpha_2 V_{10}
\end{align}

where $\alpha_1$ and $\alpha_2$ are related to the Rabi frequencies by the following expressions:

$$\alpha_j = \frac{\Lambda_j}{v}.$$  

$\alpha_1$ and $\alpha_2$ can be interpreted as the inverses of the Rabi lengths. $\Gamma$ and $g_{E_j}$ are defined as new constants:

$$\Gamma = \frac{\gamma}{v}$$

$$g_{E_j} = \frac{g_j d_j}{(c - v)v}.$$  

For simplicity of calculation and in order to obtain analytic results, we assume that the coupling field constants have the same value: $g_{E_1} = g_{E_2} = g$. 

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We have neglected the coupling between the two levels \( |1 \rangle \) and \( |2 \rangle \), so we can write at \( t = 0 \)

\[
U_{21}(0) = 0. \tag{20}
\]

At \( t = 0 \) we can suppose that the population in the upper level \( |0 \rangle \) is approximately empty and the population in level 1 and the population in level 2 are almost the same, so

\[
\rho_{01}(0) \approx 0, \quad \rho_{11}(0) \approx \rho_{22}(0) \approx \frac{1}{2}. \tag{21}
\]

The Maxwell–Bloch equations are a system of differential equations. Our interest is in studying the evolution of the fields \( \alpha_1 \) and \( \alpha_2 \). After algebraic manipulations, integration, and differentiation, we obtain a non-linear second-order system of differential equations:

\[
\begin{align*}
\frac{d^2 \alpha_1}{dz^2} &= -\frac{\alpha_1^3}{2} + \frac{g}{2} \alpha_1 + \alpha_2^2 \alpha_1 + \Gamma \frac{d \alpha_1}{dz} \\
\frac{d^2 \alpha_2}{dz^2} &= -\frac{\alpha_2^3}{2} + \frac{g}{2} \alpha_2 - \alpha_1^2 \alpha_2 + \Gamma \frac{d \alpha_2}{dz}.
\end{align*} \tag{22}
\]

The fields have slowly varying amplitudes. In this case, we can neglect the second differentiation of the fields. Thus, the system of differential equations becomes

\[
\begin{align*}
\Gamma \frac{d \alpha_1}{dz} - \Omega \alpha_1 + \frac{g}{2} \alpha_1 &= 0 \\
\frac{d \alpha_2}{dz} - \Omega \alpha_2 + \frac{g}{2} \alpha_2 &= 0
\end{align*} \tag{23}
\]

where

\[
\Omega = \alpha_1^2 + \alpha_2^2. \tag{24}
\]

\( \Omega \) is proportional to the total energy of the beam waves. It is possible to solve the above system when we have obtained the expression for \( \Omega \). So, after algebraic manipulations, we find the Bernoulli differential equation for \( \Omega \):

\[
\Gamma \frac{d \Omega}{dz} - b \Omega - \frac{b}{g} \Omega^2 = 0 \tag{25}
\]

where

\[
b = \frac{2g}{\Gamma}. \tag{26}
\]

This Bernoulli differential equation can be solved, and we obtain

\[
\Omega = \frac{1}{K \exp(bz) + \frac{2}{g}}. \tag{27}
\]

\( K \) is a positive real constant.

Now we can integrate the two decoupled differential equations:

\[
\int \frac{d \alpha_j}{\alpha_j} = \int \frac{\Omega - \frac{g}{2 \Gamma}}{\Gamma} \, dz
\]

which gives us

\[
\alpha_j = K_j \exp \left( \int \frac{\Omega - \frac{g}{2 \Gamma}}{\Gamma} \, dz \right). \tag{28}
\]

\( K_j \) are real constants.

The field amplitudes are then described by

\[
\begin{align*}
\alpha_1(z) &= \frac{K_1}{\sqrt{K \exp(bz) + \frac{2}{g}}} \\
\alpha_2(z) &= \frac{K_2}{\sqrt{K \exp(bz) + \frac{2}{g}}} \tag{29}
\end{align*}
\]

\( K_1 \) and \( K_2 \) are two free real constants, but equations (24) and (26) impose the requirement that \( K_1 \) and \( K_2 \) obey the following condition:

\[
K_1^2 + K_2^2 = 1 \tag{30}
\]

so we can write

\[
\begin{align*}
K_1 &= \cos(\varphi) \\
K_2 &= \sin(\varphi)
\end{align*}
\]

where \( \varphi \) is a free angle.

Finally, we obtain the pair of soliton shapes, described by

\[
\begin{align*}
\alpha_1(z) &= \frac{\cos(\varphi)}{\sqrt{K \exp(bz) + \frac{2}{g}}} \\
\alpha_2(z) &= \frac{\sin(\varphi)}{\sqrt{K \exp(bz) + \frac{2}{g}}} \tag{31}
\end{align*}
\]

\( \varphi \) and \( K \) can be defined as

\[
\varphi = \arctan \left( \frac{\alpha_1(0)}{\alpha_2(0)} \right) \tag{32}
\]

\[
K = \frac{1}{\alpha_1(0) + \alpha_2(0)} - \frac{2}{g} \tag{33}
\]

where

\[
\alpha_j(z = 0) = \alpha_j(0). \tag{34}
\]

4. Soliton pair properties

From equation (29), we see that we obtain four types of solution for the soliton pairs. The first type is obtained for \( 0 \leq \varphi \leq \frac{\pi}{2} \): the two field amplitudes \( \alpha_1 \) and \( \alpha_2 \) are positive (figure 2(a)).

The second type is obtained when \( \frac{\pi}{2} \leq \varphi \leq \pi \) : in this case \( \alpha_1 \leq 0 \) and \( \alpha_2 \geq 0 \) (figure 2(b)). The third type exists when \( \pi \leq \varphi \leq \frac{3\pi}{2} \) : the two amplitudes are both negative (figure 2(c)). The last type is obtained for \( \frac{3\pi}{2} \leq \varphi \leq 2\pi \) : in this case \( \alpha_1 \geq 0 \) and \( \alpha_2 \leq 0 \) (figure 2(d)).

In the first and fourth types, the two solitons propagate through the medium with the same direction of polarization, whereas in the second and third type, the two solitons propagate through the medium with contrary directions of polarization.

To study the properties of the soliton pair propagating through the medium, we calculate the values of the fields at \( \pm \infty \). These parameters help us to find some soliton properties.
Equation (32) implies that the velocity obeying the following equation:

\[ v^2 - cv + \frac{1}{2} A = 0 \]

where

\[ A = 2c V_0 \left( \frac{1}{L_1} + \frac{1}{L_2} \right) \]

allows us to conclude that there exist two possible velocities for propagation of the soliton pair: the low speed \( v_l \) and the high speed \( v_h \), given by

\[ v_l = \frac{c - \sqrt{c^2 - A}}{2} \]

\[ v_h = \frac{c + \sqrt{c^2 - A}}{2} \]  

These two last equations tell us that the soliton pair with large \( A \) propagates through the medium more slowly than the soliton pair with small \( A \).

The two velocities are obviously real, so we obtain a natural condition for existence of the soliton pair:

\[ A \leq c^2. \]  

We can rewrite this existence condition as

\[ \left( \frac{1}{L_1} + \frac{1}{L_2} \right) \leq \frac{c^2}{2g_0\Delta}. \]  

Another condition arising from the approximation of slowly varying amplitudes of the field amplitudes \( \bar{E}_1 \) and \( \bar{E}_2 \) (2) is essential. This condition is equivalent to

\[ \left| \frac{v}{\omega_j} \frac{\partial \alpha_j}{\partial z} \right| \ll |\alpha_j|, \]  

\[ \left| \frac{c}{\omega_j} \frac{\partial \alpha_j}{\partial z} \right| \ll |\alpha_j|. \]

From equations (29) we find that

\[ \left| \frac{\partial \alpha_j}{\partial z} \right| = \left( \frac{K \exp(bz)}{K \exp(bz) + \frac{b}{2}} \right) \frac{b\eta_j}{2} \ll \frac{b}{2} |\alpha_j|. \]  

Figure 2. Soliton pair shapes: \( \alpha_1 \) (full curve) and \( \alpha_2 \) (broken curve) for \( K = g = b = 1 \) and for \( \varphi = \frac{\pi}{4} \) (a), \( \varphi = \frac{2\pi}{3} \) (b), \( \varphi = \frac{3\pi}{4} \) (c), and \( \varphi = \frac{5\pi}{4} \) (d).
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So, the slowly varying amplitude condition becomes
\[
\frac{c b}{2} \ll \min(\omega_1, \omega_2).
\] (37)

5. Conclusions

In summary, we have presented analytic results that give solitary wave pairs in two-photon resonance with the medium. These solitary wave pairs can propagate through a three-level system in the lambda configuration without changing their shapes.

The two solitons have the same shapes and there are four types of soliton pair, two of which give us soliton pairs which propagate through the medium with the same direction of polarization. The other two types of soliton pair propagate through the medium with the pump field and Stokes field having opposite directions of polarization.

Moreover, we show that the soliton pair can propagate through the three-level system only when the existence condition is satisfied. Under this condition, the soliton pair propagates with two possible velocities (high and low velocity).

These results could be useful in optical data communication, where the optical fibre could be modelled as an absorbing three-level system. The advantage of the soliton in supporting data information is the invariance of the shape which minimizes the noise effect, which is usually the origin of signal defects. Also, solitons propagate without dispersion. Therefore, we can send optical information with a high bit rate.

Finally, we note that soliton pair propagation with two different velocities, in supporting data information, on the one hand, allows checking of the received data with the consequent reduction of transmission errors and, on the other hand, provides the opportunity to test or to study the evolution characteristics of the receiving system.

References

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